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A Study of Perfect Wetting for Potts and Blume–Capel Models with Correlation Inequalities

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The so-called perfect wetting phenomenon is studied for the q-state, $d \geqslant 2$ Potts model. Using a new correlation inequality, a general inequality is established for the surface tension between ordered phases ($\sigma^{a,b}$) and the surface tension between an ordered and the disordered phases ($\sigma^{a,f}$) for any even value of q. This result implies in particular $\sigma_{\beta_t}^{a,b} \geqslant \sigma_{\beta_t}^{a,f} + \sigma_{\beta_t}^{b,f} > 0$ at the transition point β_t where the previous phases coexist for q large. This inequality is connected to perfect wetting at the transition point using thermodynamic considerations. The same kinds of results are derived for the Blume–Capel model.

KEY WORDS: Perfect wetting; surface tensions; Potts model; Blume–Capel model; correlation inequalities.

1. INTRODUCTION

The coexistence of phases is a classical subject in physics, and its microscopic description is currently under progress within statistical mechanics. Belonging to the class of models which may describe the coexistence of three phases, we find the well-known Potts and Blume–Capel models.

The simple structure of the Potts model permits a rather precise analysis of its phase diagram: thus, Baxter[1] founded the exact solution at the transition temperature for two dimensions and for $q > 4$ and the existence of a latent heat at this point (i.e., a discontinuity of the energy). For $q \leqslant 4$, he got a transition of second order. In dimension 3, it is expected

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that the transition will be of first order even for $q=4$. The existence of this first-order transition has been proved rigorously by Kotecky and Shlosman[2] for q large enough and for any dimension $d \geqslant 2$. For such q, there exists a unique transition temperature $T_t = \beta_t^{-1}$ where q ordered phases $(a, b,...)$ coexist with a disordered one (f). Above β_t, only ordered phases can coexist and below β_t there is a unique phase: the disordered one. Two kinds of interfaces may therefore appear within this context: between ordered phases or between ordered and the disordered phases.

In this paper, we shall be interested in the coexistence of three phases. We are thus led to consider *a priori* two possible situations: the coexistence of three ordered phases and the coexistence of two ordered and the disordered phases. The Blume–Capel model can be studied along the same lines, since it can be shown by applying the Pirogov–Sinai theory at low temperature that there is a transition temperature where three phases can coexist.

To describe the coexistence of three phases, one needs to consider the relationship between the surface tensions that characterize the corresponding interfaces. It is known that two distinct physical situations may occur: a lens or a film of the intermediate phase. Since Gibbs already said what we wish to point out, we reproduce his own words (ref. 3, quoted before in ref. 4, from which we also take the comments inside the brackets):

> Let A, B, C, the three different fluid phases of matter, which satisfy the conditions necessary for equilibrium when they meet at plane surfaces. The components of A and B may be the same or different, but C have no components except such as belong to A or B. Let us suppose masses of the phase A and B to be separated by a very thin sheet of the phase C.... The value of the superficial tension for such a film will be $\sigma_{AC} + \sigma_{BC}$, if we denote by these symbols the tensions of the surfaces of contact of the phases A and C, and B and C, respectively.... Now if $\sigma_{AC} + \sigma_{BC}$ is greater than σ_{AB}, the tension of the ordinary surface between A and B, such a film will be at least practically unstable. [It will retract into a lens.].... We cannot suppose that $\sigma_{AB} > \sigma_{AC} + \sigma_{BC}$, for this would make the ordinary surface between A and B unstable and difficult to realize. If $\sigma_{AB} = \sigma_{AC} + \sigma_{BC}$, we may assume, in general, that this relation is not accidental, and that the ordinary surface for the contact for A and B is of the kind that we have described [that is, with A and B separated by a sheet of C].

This last situation (the film) will be defined as perfect wetting, while the previous regime (lens) will be called partial wetting.

Numerous papers have been devoted to the study of wetting (see refs. 5–7 and references therein). Within the class of models we consider, we point out several approaches which suggest the occurrence of perfect wetting at the transition point. By numerical investigations Selke[8] has shown that the net adsorption per unit length of disordered phase at the interface diverges at this particular point. The same result has been obtained by Derrida and Schick[9] in the mean field approximation. Another result which goes in the same direction has been obtained by Bricmont and Lebowitz,[10] who, using perturbative arguments for $d \geqslant 3$, study an inequality of the type considered by Gibbs for restricted surface tensions (i.e., those that take into account only the lowest energy excitations of the interface).

The aim of this paper is to study perfect wetting for the q-states Potts model and the generalized Blume–Capel model along the lines described by Gibbs. We establish the validity for any dimension $d \geqslant 2$ and any even value of q of the inequality

$$\sigma^{ab} \geqslant \sigma^{af} + \sigma^{bf} \tag{1.1}$$

which takes its full physical meaning at the transition point where σ^{ab} and σ^{af} are strictly positive.[11,12] The key of the proof is a new correlation inequality for the Potts model. In fact this correlation inequality is derived for a wider class of Hamiltonians and has its own interest (see Theorems 1–3). The same kind of inequality will also be derived for the Blume–Capel model by using FKG inequalities.[13] On the other hand, since the surface tensions between two ordered phases are equal, we have

$$\sigma^{ab} < \sigma^{ac} + \sigma^{bc} \tag{1.2}$$

from which we easily understand that an ordered phase cannot wet two other ordered ones.

Another way of describing perfect wetting is by considering the film as an infinitely flat lens with zero contact angles θ_1 and θ_2 (see Fig. 1). For

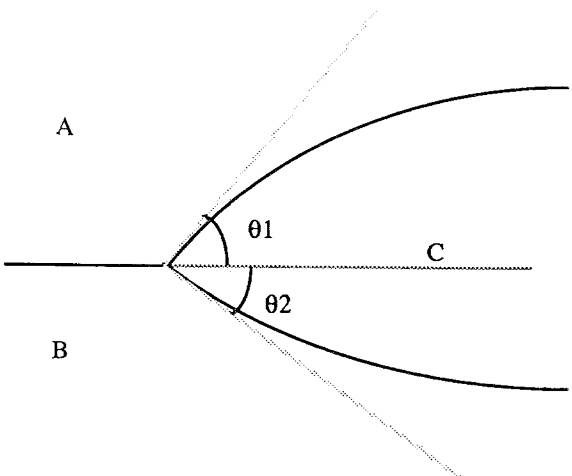

Fig. 1. Contact angles θ_1 and θ_2 for a lens of phase C.

fluid systems the contact angles of the lens of the phase C are given by the Dupré equations:

$$\sigma_{AB} = \sigma_{AC} \cos\theta_1 + \sigma_{BC} \cos\theta_2 \tag{1.3a}$$

$$0 = \sigma_{AC} \sin\theta_1 - \sigma_{BC} \sin\theta_2 \tag{1.3b}$$

and thus perfect wetting obeys the so-called Antonov rule:

$$\sigma_{AC} + \sigma_{BC} = \sigma_{AB} \tag{1.4}$$

However, for anisotropic media, the preceding relations must be corrected. A standard free energy minimization at constant volume leads in this case to the so-called Herring relations.[14] These relations are the generalizations of the Dupré equations and can be written explicitly as

$$\begin{aligned}\sigma_{AB}(0) = {} & \cos\theta_1 \sigma_{AC}(\theta_1) + \cos\theta_2 \sigma_{BC}(\theta_2)\\ & - \sin\theta_1 \sigma'_{AC}(\theta_1) - \sin\theta_2 \sigma'_{BC}(\theta_2)\end{aligned} \tag{1.5a}$$

$$\begin{aligned}\sigma'_{AB}(0) = {} & |\sin\theta_1 \sigma_{AC}(\theta_1) - \sin\theta_2 \sigma_{BC}(\theta_2)\\ & + \cos\theta_1 \sigma'_{AC}(\theta_1) - \cos\theta_2 \sigma'_{BC}(\theta_2)|\end{aligned} \tag{1.5b}$$

where $\sigma_{XY}(\theta)$ is the surface tension for an interface between X and Y, which makes an angle θ with respect to the horizontal axis, and $\sigma'_{XY}(\theta)$ is the derivative of $\sigma_{XY}(\theta)$ with respect to the angle θ. Perfect wetting still corresponds to $\theta_1 = \theta_2 = 0$ in formula (1.5). Whether or not the last relation in this case is compatible with Antonov rule will also be considered in this paper.

The paper is organized as follows. The new correlation inequalities are derived in Section 2. Section 3 is devoted to surface tension inequalities within the Potts model. The Blume–Capel model is considered in Section 4 and some concluding remarks are given in Section 5.

2. CORRELATION INEQUALITIES

We consider the q-state Potts model defined as follows: at each lattice site i of a d-dimensional cubic lattice $\mathbb{Z}^d$ there is a variable $x_i = 1, 2,..., q$; the Hamiltonian in a finite box $\Lambda \subset \mathbb{Z}^d$ is

$$H_\Lambda = -\sum_{\langle i;j \rangle \subset \Lambda} J_{ij}\delta(x_i, x_j) - \sum_{i \in \Lambda} [H_i \delta(x_i, a) + K_i \delta(x_i, b)] \tag{2.1}$$

The first sum is over nearest neighbor pairs in Λ, δ is the Kronecker delta $[\delta(x, x') = 1$ if $x = x'$ and 0 otherwise$]$, and J_{ij} are the coupling constants.

The second sum is over the sites of Λ; a and b are integers in $\{1, 2,..., q\}$; and H_i, K_i are external fields in the directions of the states a and b, respectively.

We let $\langle f \rangle_{\gamma_\Lambda}$ or $\langle f(x_i, x_j, ...) \rangle_{\gamma_\Lambda}$ denote the expectation value of a function f defined on the configurations $\{x_i\}$, $i \in \Lambda$, with respect to the Gibbs measure in Λ,

$$d\gamma_\Lambda = [Z(\Lambda)]^{-1} e^{-\beta H_\Lambda}$$

We denote by $\langle f; g \rangle_{\gamma_\Lambda} = \langle fg \rangle_{\gamma_\Lambda} - \langle f \rangle_{\gamma_\Lambda} \langle g \rangle_{\gamma_\Lambda}$ the truncated expectation value.

Theorem 1. We assume that q is an even number and the potentials are ferromagnetic: $J_{ij} \geqslant 0$, $H_i \geqslant 0$, $K_i \geqslant 0$. Then the following inequalities hold:

$$\left\langle \prod_{i \in A} \delta(x_i, a); \prod_{i \in B} \delta(x_i, a) \right\rangle_{\gamma_\Lambda} \geqslant 0 \tag{2.2a}$$

$$\left\langle \prod_{i \in A} \delta(x_i, a); \prod_{i \in B} \delta(x_i, b) \right\rangle_{\gamma_\Lambda} \leqslant 0 \tag{2.2b}$$

for any subsets A and B of Λ, and $a \neq b$.

The proof of this theorem will be based on the following discussion and we postpone it until the end of the section.

We introduce a new system on the box Λ, which will include as a particular case the above Potts model. This new system is defined as follows: to each lattice site $i \in \Lambda$ we associate the variables $\{\sigma_i, \theta_i\}$, where $\sigma_i = \pm 1$, $\theta_i \in [0, 2\pi]$. We may consider two cases; either the variable θ_i is uniformly distributed in the circle $[0, 2\pi]$ for all $i \in \Lambda$, or it takes, with the same probability, the discrete values $\theta_i = 2\pi k_i/n$, where $k_i = 0, 1,..., n-1$ for all $i \in \Lambda$.

To simplify the notations, we represent by $\int d\theta_i$ the integral $\int_0^{2\pi} d\theta_i$ in the first case and the sum $\sum_{k_i=0}^{n-1}$ in the second case.

Three families of polynomials $H_i^+(\theta_i)$, $H_i^-(\theta_i)$, and $P_{ij}(\theta_i, \theta_j)$ in the variables $e^{i\theta_i}$, $e^{i\theta_j}$ are attached respectively to each site $i \in \Lambda$ and to each pair of nearest neighbors $\langle i, j \rangle \subset \Lambda$. We assume that these polynomials are of the form

$$H_i^{\pm}(\theta_i) = \sum_{n \geqslant 0} a_n^{\pm(i)} \cos n\theta_i$$

$$P_{ij}(\theta_i, \theta_j) = \sum_{n \in Z;\, m \in Z} a_{nm}^{(ij)} \cos(n\theta_i + m\theta_j)$$

and that all coefficients $a_n^{\pm(i)}$, $a_{nm}^{(ij)}$ are nonnegative. Moreover, $P_{ij}(\theta_i, \theta_j)=0$ if $|i-j| \neq 1$. We let ρ_Λ denote the probability measure on the set of configurations $(\sigma, \theta)=(\{\sigma_i, \theta_i\}, i \in \Lambda)$ in Λ defined by

$$\rho_\Lambda(\sigma, \theta)=Z_\Lambda^{-1} \exp\left[\sum_{\langle i;j\rangle \subset \Lambda} \frac{1+\sigma_i\sigma_j}{2} P_{ij}(\theta_i, \theta_j) + \sum_{i\in\Lambda} \frac{1+\sigma_i}{2} H_i^+(\theta) + \sum_{i\in\Lambda} \frac{1-\sigma_i}{2} H_i^-(\theta)\right]$$

where Z_Λ is the normalization factor such that

$$\sum_{\sigma_i=\pm 1;\, i\in\Lambda} \int \prod_{i\in\Lambda} d\theta_i\, \rho_\Lambda(\sigma, \theta)=1$$

Theorem 2. Let μ be the following probability measure on the set of all subsets X of Λ:

$$\mu_\Lambda(X)=\int \prod_{i\in\Lambda} d\theta_i\, \rho_\Lambda(\sigma, \theta)$$

with $\sigma_i=+1$ if $i\in X$ and $\sigma_i=-1$ if $i\notin X$. Let f and g be two increasing functions defined on the subsets of Λ [i.e., such that $f(X)\geqslant f(Y)$ if $X\supset Y$]; then the inequality

$$\langle f; g\rangle_{\mu_\Lambda} \geqslant 0 \tag{2.3}$$

holds for the truncated expectation value corresponding to the measure μ_Λ.

Proof. Theorem 2 would follow from FKG inequalities provided that the inequality

$$\mu_\Lambda(X\cup Y)\, \mu_\Lambda(X\cap Y) \geqslant \mu_\Lambda(X)\, \mu_\Lambda(Y)$$

is satisfied for all X, Y contained in Λ. For each $X\subset\Lambda$ we introduce the partition function

$$Z^{P,H}(X)=\int \prod_{i\in X} d\theta_i \exp\left[\sum_{\langle i;j\rangle\subset X} P_{ij}(\theta_i, \theta_j)+\sum_{i\in X} H_i(\theta)\right]$$

When $i\in X$ and $j\in\Lambda\backslash X$ it is $1+\sigma_i\sigma_j=0$, the box Λ decomposes into the two subsets X and $\Lambda\backslash X$ without interaction between them, and we get

$$\mu_\Lambda(X)=Z^{P,H^+}(X)\, Z^{P,H^-}(X)$$

Now we have to prove that

$$Z^{P,H}(X \cup Y)\, Z^{P,H}(X \cap Y) \geqslant Z^{P,H}(X)\, Z^{P,H}(Y)$$

which follows here from Ginibre inequalities[16] by the same method as the one used in ref. 15. From this the FKG property for μ follows and Theorem 2 is proved. ▌

Theorem 3. Let A, B be subsets of Λ and let F_A, F_B be polynomials in the variables $\exp(i\theta_j)$, $j \in A$, and $\exp(i\theta_j)$, $j \in B$, respectively which may be expressed as sums of cosines with positive coefficients. Then the inequalities

$$\left\langle \prod_{i \in A} \frac{1+\sigma_i}{2} F_A ; \prod_{i \in B} \frac{1+\sigma_i}{2} F_B \right\rangle_{\rho_\Lambda} \geqslant 0 \tag{2.4a}$$

$$\left\langle \prod_{i \in A} \frac{1+\sigma_i}{2} F_A ; \prod_{i \in B} \frac{1-\sigma_i}{2} F_B \right\rangle_{\rho_\Lambda} \leqslant 0 \tag{2.4b}$$

hold for the truncated expectation value with respect to the probability measure $\rho_\Lambda(\sigma, X) \prod_{i \in \Lambda} d\theta_i$.

Proof. We assume that $A \cap B = \emptyset$, the general case being a trivial extension. We denote, for any $X \subset \Lambda$, by $\rho_X^{P,H}(\cdot)$ the expectation value with respect to the probability measure

$$d\rho_X^{P,H} = [Z^{P,H}(X)]^{-1} \exp\left[\sum_{\langle i;j \rangle \subset X} P_{ij}(\theta_i, \theta_j) + \sum_{i \in X} H_i(\theta) \right] \prod_{i \in \Lambda} d\theta_i$$

on the configurations $\{\theta_i\}$, $i \in X$, and introduce the functions

$$f_A(X) = \prod_{i \in A} \frac{1+\sigma_i}{2}\, \rho_X^{J,H^+}(F_A)$$

$$g_B(X) = \prod_{i \in B} \frac{1-\sigma_i}{2}\, \rho_{\Lambda \setminus X}^{J,H^-}(F_B)$$

We observe that $f_A(X) = 0$ unless $X \supset A$, that $g_B(X) = 0$ unless $\Lambda \setminus X \supset B$, and, by computing first the integration with respect to the variables θ_i, $i \in \Lambda$, that

$$\left\langle \prod_{i \in A} \frac{1+\sigma_i}{2} F_A \right\rangle_{\rho_\Lambda} = \sum_{X \subset \Lambda} \mu_\Lambda(X) f_A(X)$$

Moreover,

$$\left\langle \prod_{i \in A} \frac{1+\sigma_i}{2} F_A ; \prod_{i \in B} \frac{1-\sigma_i}{2} F_B \right\rangle_{\rho_\Lambda} = \langle f_A ; g_B \rangle_{\mu_\Lambda}$$

But from Ginibre inequalities it follows that

$$\rho_{X_1}^{J,H^+}(F_A) \geqslant \rho_{X_2}^{J,H}(F_B)$$

when $X_1 \supset X_2 \supset A$. Hence, $f_A(X)$ is an increasing function on the subsets of Λ. Again Ginibre inequalities prove that $g_B(X)$ is a decreasing function. Then Theorem 2 implies that $\langle f_A; g_B \rangle_{\mu_\Lambda} \leqslant 0$, which proves the second inequality in Theorem 3. The first inequality can be proved by the same method. ∎

Proof of Theorem 1. The inequality stated in the theorem for the Potts model is a particular case of those of Theorem 3. In order to see this, we establish the following correspondence between the configurations $x_i \in \{1,..., q\}$ with $q = 2n$ and the configurations (σ_i, θ_i) with $\sigma_i = \pm 1$, $\theta_i = 2\pi k_i/n$, and $k_i = 0, 1,..., n-1$: for $1 \leqslant x_i \leqslant n$ we take $\sigma+1$ and $\theta_i = 2\pi(x_i - 1)/n$, and for $n+1 \leqslant x_i \leqslant 2n$ we take $\sigma_i = -1$ and $\theta_i = 2\pi(x_i - n - 1)/n$. We choose $a = 1$, $b = n+1$. Then

$$\delta(x_i, x_j) = \frac{1 + \sigma_i \sigma_j}{2} \delta_0(\theta_i - \theta_j)$$

$$\delta(x_i, a) = \frac{1 + \sigma_i}{2} \delta_0(\theta_i)$$

$$\delta(x_i, b) = \frac{1 - \sigma_i}{2} \delta_0(\theta_i)$$

where $\delta_0(0) = 1$ and $\delta_0(\theta) = 0$ if $\theta \neq 0$. Since $\delta_0(\theta)$ may be written as

$$\delta_0(\theta) = \frac{1}{n} + \frac{1}{n} \sum_{m=1}^{n-1} \cos m\theta$$

the hypotheses of Theorem 3 are satisfied. ∎

3. WETTING IN THE POTTS MODEL

In order to define the surface tensions mentioned previously, let us consider the Potts model with zero external field and different boundary conditions. The Hamiltonian in a finite box Λ for a configuration $x_\Lambda \equiv \{x_i\}$, $i \in \Lambda$, with boundary condition $\tilde{x}$ (i.e., a configuration on $\mathbb{Z}^d$) is

$$H_\Lambda^{\text{b.c.}}(x_\Lambda \mid \tilde{x}) = - \sum_{\langle i;j \rangle \cap \Lambda \neq \varnothing} \delta(x_i, x_j) \tag{3.1}$$

where $x_i = \tilde{x}_i$ if $i \notin \Lambda$, and we denote by $\langle \cdot \rangle_\Lambda^{\text{b.c.}}$ the expectation corresponding to the Gibbs measure in Λ, $[Z_\Lambda^{\text{b.c.}}(\beta)]^{-1} \exp(-\beta H_\Lambda^{\text{b.c.}})$.

In the following, we focus on the following types of boundary conditions:

(a) The ordered b.c. obtained by fixing $\tilde{x}_i = a$, $a = 1, 2,..., q$, for every i in $\mathbb{Z}^d$.

(f) The free b.c. (f), where the sum in (3.1) runs only over the pairs $\langle i; j\rangle$ included in Λ.

($a, b, \mathbf{n}$) With respect to an arbitrary plane defined by a d vector $\mathbf{n}$, the mixed ($a, b, \mathbf{n}$) b.c. is defined by

$$\begin{aligned} \tilde{x}_i &= a \qquad \text{if} \quad i_1 n_1 + \cdots + i_d n_d \geqslant 0 \\ \tilde{x}_i &= b \qquad \text{if} \quad i_1 n_1 + \cdots + i_d n_d < 0 \end{aligned}$$

($a, f, \mathbf{n}$) With respect to an arbitrary plane characterized by a d vector $\mathbf{n}$, the mixed ($a, f, \mathbf{n}$) b.c. is defined by putting $\tilde{x}_i = a$ above the plane and taking the free b.c. below it.

We are now able to define the surface tensions in our problem. However, since the model is anisotropic, we also have to take into account the angular dependence of these surface tensions. We consider a rectangular box Λ centered at the origin. Let $S_\Lambda(\mathbf{n})$ be the area of the portion of the plane defined by $\mathbf{n}$ inside Λ. We define the surface tension at the inverse temperature β by

$$\sigma_\beta^{a,b}(\mathbf{n}) = \lim_{\Lambda \uparrow Z^d} -\frac{1}{S_\Lambda(\mathbf{n})} \log \frac{Z_\Lambda^{a,b,\mathbf{n}}(\beta)}{[Z_\Lambda^a(\beta)\, Z_\Lambda^b(\beta)]^{1/2}} \tag{3.2}$$

To simplify our notations, we also introduce

$$\begin{aligned} \sigma_\beta^{a,b} &\equiv \sigma_\beta^{a,b}(\mathbf{n}) \qquad \text{for} \quad \mathbf{n} = (1, 0,..., 0) \\ \sigma_\beta^{a,b}(\theta) &\equiv \sigma_\beta^{a,b}(\mathbf{n}) \qquad \text{for} \quad \mathbf{n} = (\cos\theta, \sin\theta, 0,..., 0) \end{aligned}$$

Remark 2. Definition (3.2) is justified by noticing that in this expression the volume terms proportional to the free energy of the coexisting phases as well as the terms corresponding to the boundary effects cancel and only the term that takes into account the free energy of the interface is left; see, for example, refs. 17–19. The limit (3.2) has been proved to exist in ref. 22 when a and b correspond to two ordered phases and $\mathbf{n} = (1, 0,..., 0)$; the same result is not yet known when a is an ordered phase and $b = f$.

Remark 2. To be complete, one should also point out that formula (3.2) for the (a, f) b.c. gives indeed an interface free energy between two coexisting pure phases. Although for the Ising model, the combination of

$+$ or $-$ b.c. with free b.c. will generate inside all the volume the $+$ or $-$ phase and hence $\sigma_\beta^{a,f}=0$ for any β, this is not the case for the Potts model at β_t and large q. In fact, the state that gives equal probability to all configurations (free b.c.) is a "ground state" obtained in the limit when $q\to\infty$ and $\beta=\beta_t\approx(1/d)\log q$. The other q ground states at this point are the Dirac measures on the completely ordered configurations $x_i=a$ for all $i\in\mathbb{Z}^d$ and $a\in\{1,\dots,q\}$. The fact that with free b.c. one obtains the disordered pure phase may be proved as follows. We first notice that the even correlation functions with free b.c. are extremal among the translation-invariant states. This may be derived using the inequality (4.22) of ref. 20. Since there are only $q+1$ states,[21] we can deduce the unicity of the disordered state, which is therefore extremal.

Due to the symmetry of the model, $\sigma_\beta^{a,b}(\mathbf{n})$ has the same value for every choice of a and b. Analogously, all the $\sigma_\beta^{a,f}(\mathbf{n})$ are equal, independently of a. For such tensions, we have the following theorem.

Theorem 4. For the q-state Potts model with q even and $d\geqslant 2$, we have

$$\sigma_\beta^{a,b}(\mathbf{n})\geqslant\sigma_\beta^{a,f}(\mathbf{n})+\sigma_\beta^{b,f}(\mathbf{n}) \tag{3.3}$$

for any temperature β^{-1} for any orientation $\mathbf{n}$, and for any two different ordered phases a and b.

Proof. By definition of the surface tensions a sufficient condition to ensure the validity of (3.3) is

$$\frac{Z_\Lambda^{a,b,\mathbf{n}}(\beta)}{[Z_\Lambda^a(\beta)\,Z_\Lambda^b(\beta)]^{1/2}}\leqslant\frac{Z_\Lambda^{a,f,\mathbf{n}}(\beta)}{[Z_\Lambda^a(\beta)\,Z_\Lambda^f(\beta)]^{1/2}}\,\frac{Z_\Lambda^{b,f,\mathbf{n}}(\beta)}{[Z_\Lambda^b(\beta)\,Z_\Lambda^f(\beta)]^{1/2}}$$

This inequality can also be written as

$$\frac{Z_\Lambda^{a,b,\mathbf{n}}(\beta)}{Z_\Lambda^f(\beta)}\leqslant\frac{Z_\Lambda^{a,f,\mathbf{n}}(\beta)}{Z_\Lambda^f(\beta)}\,\frac{Z_\Lambda^{b,f,\mathbf{n}}(\beta)}{Z_\Lambda^f(\beta)}$$

Due to the symmetry of the partition functions with respect to the plane defining the interface, this last inequality is in fact a straightforward consequence of Theorem 2, namely

$$\begin{aligned}&\Big\langle\prod_{i\in\partial\Lambda^+}\delta(x_i,a)\prod_{i\in\partial\Lambda^-}\delta(x_i,b)\Big\rangle_\Lambda^f(\beta)\\&\quad\leqslant\Big\langle\prod_{i\in\partial\Lambda^+}\delta(x_i,a)\Big\rangle_\Lambda^f(\beta)\Big\langle\prod_{i\in\partial\Lambda^-}\delta(x_i,b)\Big\rangle_\Lambda^f(\beta)\end{aligned}$$

where $\partial\Lambda^+$ (resp. $\partial\Lambda^-$) is the part of the boundary located above (resp. below) the plane.

In fact this inequality implies the previous one if the definition of surface tension is slightly modified in the following sense. In the region where the boundary condition is called free one needs an extra layer of spins which may assume one of the q values and interact with the spins in Λ. However, if we want to keep the previous definition of the surface tensions, the following expansion may be used:

$$\prod_{i \in C} e^{\beta\delta(x_i, a)} = \sum_{X \subset C} (e^\beta - 1)^{|X|} \prod_{i \in X} \delta(x_i, a)$$

and the inequality of Theorem 4 is a consequence of Theorem 1 when A runs over the subsets of $\partial\Lambda^+$ and B over the subsets of $\partial\Lambda^-$. ∎

For completeness, we collect in the following theorem some properties of the surface tensions that in part have been proved previously.

Theorem 5. For Potts models with q even and large enough and if $d=2$, we have

$$\sigma_\beta^{a,f} = 0 \qquad \text{for any} \quad \beta \neq \beta_t \tag{3.4}$$

$$\sigma_{\beta_t}^{a,f} > 0 \tag{3.5}$$

$$\sigma_\beta^{a,b} > 0 \qquad \text{for} \quad \beta \geqslant \beta_t \tag{3.6}$$

$$\sigma_\beta^{a,b} = 0 \qquad \text{for} \quad \beta < \beta_t \tag{3.7}$$

Moreover, for $d=3$, relations (3.5)–(3.7) also hold and $\sigma_\beta^{a,f}=0$ for $\beta < \beta_t$ [we expect (3.4) to be true for any $\beta \neq \beta_t$].

Proof. Relations (3.5)–(3.7) have been proved in refs. 11 and 22. The proof of relation (3.4) proceeds as follows. For $\beta < \beta_t$, the use of inequality (3.3) leads to

$$\sigma_\beta^{a,f} = 0$$

On the other hand, since the model is self-dual for $d=2$, we have by symmetry

$$\sigma_\beta^{a,f} = \sigma_{\beta^*}^{a,f}$$

with β^* defined by

$$(e^\beta - 1)(e^{\beta^*} - 1) = q$$

From this last relation, it is easily seen that if $\beta < \beta_t = \log(\sqrt{q}+1)$, we have $\beta^* > \beta_t$ and therefore

$$\sigma_\beta^{a,f} = 0 \qquad \text{for any} \quad \beta \neq \beta_t$$

For $d=3$, the proofs may be found in refs. 12 and 22. The fact that $\sigma_\beta^{a,f}=0$ for $\beta < \beta_t$ may be deduced from (3.3) and (3.7). ∎

Remark 3. Within the Potts model, surface tensions between ordered phases are increasing with respect to the inverse temperature β. This can be seen by first deriving with respect to β the partition functions appearing in the definition of the surface tensions. The result can then be written as the difference of two correlation functions with different boundary conditions. That this difference has a definite sign is implied by inequalities (2.2).

Combining Theorems 4 and 5, we easily get the following proposition.

Corollary 1. For $d=2$, $d=3$ Potts models (with q even and large enough), we have

$$\sigma_{\beta_t}^{a,b} \geqslant \sigma_{\beta_t}^{a,f} + \sigma_{\beta_t}^{b,f} > 0 \tag{3.8}$$

at the transition temperature.

Remark 4. By using the duality symmetry and correlation inequalities it has been proved in ref. 22 that for $d=2$ the surface tension $\sigma_\beta^{a,b}$ is equal to the inverse correlation length (mass) of the two-point correlation function $\langle \delta(x_i, x_j) \rangle$ at the dual temperature β^*. Then inequality (3.8) implies the existence of a mass gap at β_t for this model and gives an alternative proof of this result already known from ref. 11.

According to Remark 3 and Corollary 1, we therefore expect for the surface tensions the behavior plotted in Fig. 2.

In the graph of $\sigma_\beta^{a,b}$, we have drawn a vertical tangent at β_t where $\sigma_\beta^{a,b}$ is discontinuous. Modulo some limit permutations, this corresponds to the divergence for the quantity E defined by

$$E = \frac{1}{S_A(\mathbf{n})} \sum_{\langle i;j \rangle \subset A} [\langle \delta(x_i, x_j) \rangle^{a,b}(\beta) - \langle \delta(x_i, x_j) \rangle^{a}(\beta)]$$

which may be interpreted as a width for the interface analogous to the width studied numerically by others[8]:

$$W = \frac{1}{L} \sum_{i \in A;\, n \neq a;\, b} [\langle \delta(x_i, n) \rangle^{a,b} - \langle \delta(x_i, n) \rangle^{a}]$$

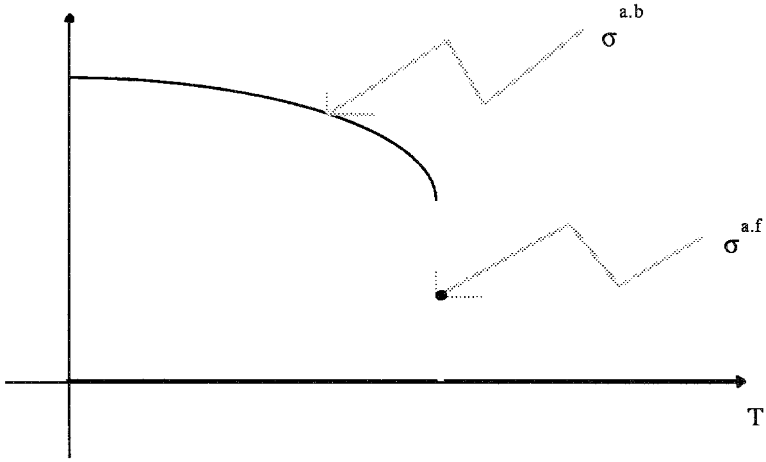

Fig. 2. Surface tensions $\sigma^{a,b}$ and $\sigma^{a,f}$ as functions of $T=\beta^{-1}$.

The characteristics of this divergence will not be further analyzed in this paper.

The physical interest of Corollary 1 is contained in the beautiful Gibbs remark quoted in the introduction. Indeed, this leads us to believe that the equality

$$\sigma_{\beta_t}^{a,b}=\sigma_{\beta_t}^{a,f}+\sigma_{\beta_t}^{b,f} \tag{3.9}$$

between surface tensions is satisfied and that there is a sheet of disordered phase f separating the two ordered phases a and b at the transition temperature. This is perfect wetting.

One may then wonder if this conclusion is in agreement with the Herring relations which describe the perfect wetting:

$$\sigma_{\beta}^{a,b}=\sigma_{\beta}^{a,f}+\sigma_{\beta}^{b,f} \tag{3.10}$$

$$(\sigma_{\beta}^{a,b})'=|(\sigma_{\beta}^{a,f})'-(\sigma_{\beta}^{b,f})'| \tag{3.11}$$

where the prime indicates the derivative with respect to the angle.

Equation (3.10) should indeed be satisfied at β_t as indicated previously. The second condition reduces trivially to $0=0$ for the Potts model whenever the derivatives indeed exist, since $\sigma_{\beta}^{a,b}(\theta)$ and $\sigma_{\beta}^{a,f}(\theta)$ are even function of θ. When one of the derivatives does not exist at β_t we expect that all three derivatives that appear in (3.11) do not exist either; this is a consequence of inequality (3.3) and Gibbs' argument, which implies the equality [i.e., formula (3.9) would be valid for any angle θ].

However, in such a case, Herring[14] pointed out that it is enough to satisfy (3.11) for one admissible value to get a stable physical situation; by admissible value we mean any value between the left and the right derivatives.

We expect the trivial case to occur always for $d=2$. However, for $d \geqslant 3$ and for q large enough, one should be able to prove that the interface between a and f for $\beta=\beta_t$ is rigid by using, for instance, Dobrushin method.[23] The rigidity of an interface is indeed believed to imply the non-derivability of the corresponding surface tension. This is due to the appearance of a facet which is characterized by a cusp in the Wulff plot.

From this discussion we see that there is no disagreement within the Potts model for perfect wetting between the Antonov rule and the Herring relations.

4. THE BLUME–CAPEL MODEL

We consider a system of spins s_i which take the values $+1$, 0, -1. The generalized Blume–Capel Hamiltonian is of the form

$$H_\Lambda(s_1 \cdots s_\Lambda) = J \sum_{\langle i;j\rangle \subset \Lambda} |s_i - s_j|^\gamma - \mu \sum_{i \in \Lambda} s_i^2 \tag{4.1}$$

where $\gamma > 1$, $\mu \geqslant 0$, and $J \geqslant 0$. This Hamiltonian has three ground states at $\mu = 0$ (the three configurations where all the spins take the same value). At low enough temperature and for $d \geqslant 2$ the phase diagram of the model is as follows[24]: there exists a first-order transition line $\beta_t(\mu)$ on which three phases coexist (obtained as thermodynamic limit of finite-volume Gibbs states with the $+1$, 0, or -1 boundary conditions). Above this line two phases coexist, obtained respectively with the $+1$ and -1 b.c.; below this line there is only one phase obtained with the 0 b.c.

The inequality

$$\left\langle \prod_{i \in \partial\Lambda^+} \delta(s_i, +1); \prod_{i \in \partial\Lambda^-} \delta(s_i, -1) \right\rangle \leqslant 0 \tag{4.2}$$

holds because the model satisfies FKG inequalities with respect to the partial order induced in the configurations by $s_i > s_i'$ for every i, and $\delta(s_i, 1)$ [resp. $\delta(s_i, -1)$] is an increasing (resp. a decreasing) function of the configurations. Therefore, arguing as in the preceding section, it follows that

$$\sigma_\beta^{+,-} \geqslant \sigma_\beta^{+,0} + \sigma_\beta^{-,0} \tag{4.3}$$

with obvious notations. Using again the Gibbs argument, we can also give here the same kinds of physical considerations as in Section 3 for the sheet of 0 phase which wets perfectly the + and − phases.

5. CONCLUDING REMARKS

In this paper, we have proved inequalities between surface tensions by using correlation inequalities and in particular

$$\sigma_{\beta_t}^{a,b} \geqslant \sigma_{\beta_t}^{a,f} + \sigma_{\beta_t}^{b,f} > 0$$

This inequality together with the Gibbs arguments leads to perfect wetting in the Potts and Blume–Capel models at the transition temperature.

It would certainly be interesting to extend our analysis to a broader class of models. Moreover, to get a completely rigorous picture of perfect wetting one would need to prove the expected equality relating the surface tensions using statistical mechanical arguments. Another interesting effort would be to study the thickness of the intermediate layer.

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REFERENCES

1. R. J. Baxter, *J. Phys. C: Solid State Phys.* **6**:L445 (1973); *J. Phys. A: Math. Gen.* **15**:3329 (1982).
2. R. Kotecky and S. B. Shlosman, *Commun. Math. Phys.* **83**:493 (1982).
3. J. W. Gibbs, *The Collected Works of J. Willard Gibbs*, Vol. 1 (Longmans, Green, New York, 1928).
4. J. S. Rowlinson and B. Widom, *Molecular Theory of Capillarity* (Clarendon, Oxford, 1982).
5. M. Fisher, *J. Stat. Phys.* **34**:667 (1984).
6. J. De Coninck and F. Dunlop, *J. Stat. Phys.* **47**:827 (1987).
7. C. E. Pfister and J. Fröhlich, *Commun. Math. Phys.* **109**:493 (1987).
8. W. Selke, in *Static Critical Phenomena in Inhomogeneous Systems*, A. Pekalsky and J. Sznajd, eds. (Springer, Berlin, 1984).
9. B. Derrida and M. Schick, *J. Phys. A: Math. Gen.* **19**:1439 (1986).
10. J. Bricmont and J. L. Lebowitz, *J. Stat. Phys.* **46**:1015 (1987).
11. L. Laanait, A. Messager, and J. Ruiz, *Commun. Math. Phys.* **105**:527 (1986).
12. R. Kotecky, L. Laanait, A. Messager, and J. Ruiz, The q-state Potts model in the standard Pirogov–Sinai theory, preprint CPT-86/P.1948 (1986).
13. C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Commun. Math. Phys.* **22**:89 (1971).
14. C. Herring, in *Structure Properties of Solid Surfaces*, R. Gomer and C. S. Smith, eds. (University of Chicago Press, 1953).

15. J. L. Lebowitz, *Commun. Math. Phys.* **35**:87 (1974).
16. J. Ginibre, *Commun. Math. Phys.* **16**:310 (1970).
17. G. Gallavotti, A. Martin Löf, and S. Miracle-Solé, in *Statistical Mechanics and Mathematical Problems* (Springer, Berlin, 1973).
18. J. Bricmont, J. L. Lebowitz, and A. Messager, *Phys. Lett. A* **95**:169 (1983).
19. C. E. Pfister, in *Scaling and Self-Similarity in Physics*, J. Fröhlich, eds. (Birkhaüser, Boston, 1983).
20. C. E. Pfister, *Commun. Math. Phys.* **86**:375 (1982).
21. D. H. Martirosian, *Commun. Math. Phys.* **105**:281 (1986).
22. L. Laanait, *Phys. Lett. A* **124**:480 (1987).
23. R. L. Dobrushin, *Theor. Prob. Appl.* **17**:582 (1972).
24. J. Slawny, in *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, 1986).